A Construction of the Null Set

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Abstract

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1 Introduction

The imaginary number i satisfies the relation $i^2 = -1$, but this information falls short of a construction. Accordingly, one constructs i by identifying it with the coset $X + (X^2 + 1)$ of the ideal $(X^2 + 1)$ in $\mathbb{R}[X] / (X^2 + 1)$, where $\mathbb{R}[X]$ is the ring of real polynomials in X. This produces a natural injection of the reals, extending them to include i; this injection maps each real r to the element $r + (X^2 + 1)$, the coset of r in $\mathbb{R}[X]$. The purely imaginary numbers of the form ci correspond to the coset $cX + \{X^2 + 1\}$, and thus each complex number $r + ci$ corresponds to the coset $r + cX + \{X^2 + 1\}$. This fully constructs the complex numbers as an extension of the reals.

It seems that defining the null set, say, as the unique set that is a subset of every set, or the set that has no members, also falls short of a construction.

In the following, we propose a rigorous construction. To begin, let us assume that the null set is undefined in our set theory. We will show that this set theory may be extended uniquely to one in which the null set is defined.

2 An Induced Boolean Algebra

Let U be any set with at least two distinct members. We begin with

$P(U) \equiv \{\{S_1, S_2\}\,; S_1, S_2 \subseteq U\}$, and define an equivalence relation on $P(U)$. Indeed, we say that for any $\{A, B\}$ and $\{C, D\}$ in $P(U)$,

$\{A, B\} \equiv \{C, D\}$ if and only if $x \in A$ and $x \in B \Leftrightarrow x \in C$ and $x \in D$. To put it another way, either A and B have no members in common and neither do C and D, or A and B have members in common, C and D have members in common, and $A \cap B = C \cap D$.

Let $W(U) \equiv \left\{\overline{\{X, Y\}}; X, Y \subseteq U\right\}$, where $\overline{\{X, Y\}}$ denotes the equivalence class containing the pair $\{X, Y\}$. $W(U)$ is the collection of equivalence classes of $P(U)$ under the equivalence relation $\equiv$. We denote $\bar{\emptyset} \equiv \overline{\{S_1, S_2\}}$ for every $S_1, S_2 \subseteq U$ such that S_1 and S_2 have no members in common.

Next, we define binary operators $\sqcap$, $\sqcup$ and $\sqsubseteq$ and the unary operator $\bar{c}$ on $W(U)$:

1. For any $S_1, S_2 \subseteq U$, $\overline{\{S_1, S_1\}} \sqcap \overline{\{S_2, S_2\}} \equiv \overline{\{S_1, S_2\}}$. If S_1 and S_2 have elements in common, then $\overline{\{S_1, S_1\}} \sqcap \overline{\{S_2, S_2\}} \equiv \overline{\{S_1 \cap S_2, S_1 \cap S_2\}}$, and $\overline{\{S_1, S_1\}} \sqcap \overline{\{S_2, S_2\}} \equiv \bar{\emptyset}$ otherwise.

2. $\overline{\{S_1, S_1\}} \sqcap \overline{\{S_2, S_2\}} \equiv \overline{\{S_1, S_2\}}$ for any $S_1, S_2 \subseteq U$ with elements in

common, and $\overline{\{S_1, S_1\}} \sqcap \overline{\{S_2, S_2\}} \equiv \bar{\emptyset}$ otherwise.

3. $\overline{\{S, S\}} \sqcap \bar{\emptyset} \equiv \bar{\emptyset}$ and $\overline{\{S, S\}} \sqcup \bar{\emptyset} \equiv \bar{\emptyset}$ for any $S \subseteq U$.

4. $\overline{\{S_1, S_1\}} \sqcup \overline{\{S_2, S_2\}} \equiv \overline{\{S_1 \cup S_2, S_1 \cup S_2\}}$ for any $S_1, S_2 \subseteq U$; in particular, $\overline{\{S_1, S_1\}} \sqcup \overline{\{U, U\}} \equiv \overline{\{U, U\}}$

5. $\overline{\{S, S\}}^{\bar{c}} \equiv \overline{\{S^c, S^c\}}$ for any $S \subsetneq U$, and $\overline{\{U, U\}}^{\bar{c}} \equiv \bar{\emptyset}$.

6. $\overline{\{S_1, S_1\}} \sqsubseteq \overline{\{S_2, S_2\}} \Leftrightarrow S_1 \subseteq S_2$ for any $S_1, S_2 \subseteq U$; we stipulate $\bar{\emptyset} \sqsubseteq \overline{\{S, S\}}$ for any $S \subseteq U$, and $\bar{\emptyset} \sqsubseteq \bar{\emptyset}$.

We now observe that we can embed every subset S containing at least one member into the collection of equivalence classes as follows:

Recall $X(U) \equiv \left\{\overline{\{X, Y\}}; X, Y \subseteq U\right\}$. We define the mapping

$f : 2^U \to X(U)$ verifying $f(S) \equiv \overline{\{S, S\}}$. This mapping is clearly surjective; it is also injective on the collection of pairs of sets with at least one member in common. Indeed, suppose $S_1 \cap S_2 \neq \emptyset$ and $f\{S_1\} = f\{S_2\}$; then $\overline{\{S_1, S_1\}} = \overline{\{S_2, S_2\}}$, thus $\{S_1, S_1\} \equiv \{S_2, S_2\}$, which gives $S_1 \cap S_1 = S_2 \cap S_2$, i.e. $S_1 = S_2$. The remaining equivalence class, the one corresponding to pairs of sets with no elements in common is then defined to represent the empty set; we denote this equivalence class by $\bar{\emptyset}$. Next, we define the analogues of basic set operations on members of $X(U)$ so that f respects them.

f is then a bijection that satisfies

1. $f(X \cap Y) \equiv f(X) \sqcap f(Y)$ for any $X, Y \in$ with elements in common, and $\overline{\{S_1, S_1\}} \sqcap \overline{\{S_2, S_2\}} \equiv \overline{\{\emptyset, \emptyset\}}$ otherwise.

2. $\overline{\{S,S\}} \sqcap \overline{\{\emptyset,\emptyset\}} \equiv \overline{\{\emptyset,\emptyset\}}$ and $\overline{\{S,S\}} \sqcup \overline{\{\emptyset,\emptyset\}} \equiv \overline{\{S,S\}}$ for any nonempty S.

3. $\overline{\{S_1,S_1\}} \sqcup \overline{\{S_2,S_2\}} \equiv \overline{\{S_1 \cup S_2, S_1 \cup S_2\}}$ for any nonempty S_1, S_2; in particular, $\overline{\{S_1,S_1\}} \sqcup \overline{\{U,U\}} \equiv \overline{\{U,U\}}$

4. $\overline{\{S,S\}}^{\bar{c}} \equiv \overline{\{S^c,S^c\}}$ for any nonempty $S \neq U$, and $\overline{\{U,U\}}^{\bar{c}} \equiv \overline{\{\emptyset,\emptyset\}}$.

5. $\overline{\{S_1,S_1\}} \sqsubseteq \overline{\{S_2,S_2\}} \Leftrightarrow S_1 \subseteq S_2$ for any nonempty S_1, S_2; we stipulate $\overline{\{\emptyset,\emptyset\}} \sqsubseteq \overline{\{S,S\}}$ for any nonempty S, and $\overline{\{\emptyset,\emptyset\}} \subseteq \overline{\{\emptyset,\emptyset\}}$.

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